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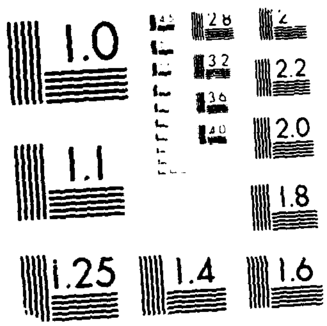


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# ADMISSIBLE BAYES TESTS FOR STRUCTURAL RELATIONSHIP

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**Abstract**

It is an open problem to construct a test for structural relationship among the mean vectors of several multivariate normal populations with known but unequal covariance matrices. In this paper, a class of admissible Bayes tests for the above problem is derived. As a byproduct, in the special case of known and equal covariance matrices, the likelihood ratio test of Rao(1973) is shown to be admissible Bayes.

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*Key words and phrases.* Structural relationship, admissible Bayes test, likelihood ratio test, prior distribution.

**1. Introduction.** Fisher(1939) pointed out the importance of tests for structural relationship among the mean vectors of several multivariate normal populations. To be specific, consider $k$ independent $p$-variate multinormal populations, $N_p(\mu_i, \Sigma_i)$, $i = 1, \cdots, k$ and the problem of testing

$$(1.1) \qquad \begin{aligned} H_0 &: \quad H\mu_i = \xi\,, i = 1, \cdots, k \qquad \textit{versus} \\ H_1 &: \quad \mu_1, \cdots, \mu_k \ \textit{arbitrary} \end{aligned}$$

where $H : s \times p$ is an unknown matrix of known rank $s < p$ and $\xi$ is an unknown $s$-vector. The hypothesis $H_0$, if true, implies that the $k$ mean vectors $\mu_1, \cdots, \mu_k$ lie in a $(p - s)$-dimensional subspace of the Euclidean $p$-space $E^p$ rather than in $E^p$ itself, and provides a structural relationship among the mean vectors. As Fisher(1939) and Rao(1973) rightly emphasized, it is essential that in dealing with several multinormal populations a hypothesis of the form (1.1) be tested prior to using the models for prediction etc.

It is an open problem in the literature to construct a suitable test for the hypotheses in (1.1) based on independent samples from the $k$ populations when $\Sigma_1, \cdots, \Sigma_k$ are unequal. In the case when $\Sigma_1, \cdots, \Sigma_k$ are equal (and $=$ $\Sigma$, say), Rao(1973) derived the likelihood ratio test (LRT) of $H_0$ against $H_1$

which rejects the null hypothesis for large values of $T = \sum_{i=1}^{s} \lambda_i(B\Sigma^{-1})$. Here $B$ is the matrix of the sums of squares and products due to the hypotheses (between groups), $\lambda_1 < \lambda_2 < \cdots < \lambda_p$ are the ordered roots of $B\Sigma^{-1}$, and the natrix $\Sigma$ is assumed known. If $\Sigma$ is unknown, one replaces $\Sigma$ in the definition of $T$ by $W$, the within matrix sum of squares and products due to error (within groups). That the resultant test is the LRT is proved in Fujikoshi(1974) and also in Rao(1985).

It is the object of this paper to provide a simple solution to the above open problem when $\Sigma_1, \cdots, \Sigma_k$ are assumed known. We have derived in *Section 2 a class of admissible Bayes tests for the hypothesis* $H_0$ versus $H_1$. Interestingly enough, inspite of the somewhat complicated structures of the model and the problem, the derived tests are of extremely simple form. In the special case of equal and known covariance matrices, the LRT of Rao(1973) is shown to be admissible Bayes.

We may recall that a Bayes critical region (for 0–1 loss function) is of the form

$$\{\mathbf{x} : \int f(\mathbf{x};\theta)\pi_1(d\theta) / \int f(\mathbf{x};\theta)\pi_0(d\theta) > c\} \tag{1.2}$$

for some positive constant c, where $f(\mathbf{x};\theta)$ is the underlying joint density,

$\theta$ is the vector of parameters, $\pi_1$ and $\pi_0$ are the prior probability measures over the alternative parameter space $\Theta_1$ and the null parameter space $\Theta_0$ respectively, and $\int$ is over the respective parameter spaces (Kiefer and Schwartz, 1965). Assuming that we have available a random sample of size $n_i$ from the *ith* population, denoted as $\mathbf{x}_{i1}, \cdots, \mathbf{x}_{in_i}$, $i = 1, \cdots, k$, we can write

$$
\begin{aligned}
(1.3) \quad f(\mathbf{x};\theta) \;&=\; (2\pi)^{-np/2} \prod_{i=1}^{k} |\Sigma_i|^{-n_i/2} etr\{-1/2 \sum_{i=1}^{k} n_i \Sigma_i^{-1} (\bar{\mathbf{x}}_i - \mu_i) \\
&\qquad \cdot (\bar{\mathbf{x}}_i - \mu_i)'\} etr\{-1/2 \sum_{i=1}^{k} \Sigma_i^{-1} S_i\}
\end{aligned}
$$

where $\bar{\mathbf{x}}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} \mathbf{x}_{ij}$, $S_i = \sum_{j=1}^{n_i} (\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)(\mathbf{x}_{ij} - \bar{\mathbf{x}}_i)'$, $i = 1, \cdots, k$,

$n = \sum_{i=1}^{k} n_i$, $\theta = (\mu_1, \cdots, \mu_k)$, $\Theta = \{(\mu_1, \cdots, \mu_k) : \mu_i \in E^p, \; i = 1, \cdots, k\}$,

$\Theta_0 = \{(\mu_1, \cdots, \mu_k) : H\mu_i = \xi, \; i = 1, \cdots, k, \; H : s \times p \; unknown,$

$rank(H) = s < p, \; s \; known, \; \xi : s \times 1 \in E^p \; unknown\}$, $\Theta_1 = \Theta - \Theta_0$.

In what follows the factor $(2\pi)^{-np/2} \prod_{i=1}^{k} |\Sigma_i|^{-n_i/2} etr\{-1/2 \sum_{i=1}^{k} \Sigma_i^{-1} S_i\}$ appearing in $f(\mathbf{x};\theta)$ is ignored because it is independent of $\theta$ and has no influence on Bayes tests of the form (1.2). Thus one can write (1.2) in the form

$$
(1.4) \qquad \{\mathbf{x} : \int f^*(\mathbf{x};\theta)\pi_1(d\theta) / \int f^*(\mathbf{x};\theta)\pi_0(d\theta) > c\}
$$

where

$$(1.5)\qquad f^*(\mathbf{x};\theta) = etr\{-1/2\sum_{i=1}^{k} n_i\Sigma_i^{-1}(\bar{\mathbf{x}}_i - \mu_i)(\bar{\mathbf{x}}_i - \mu_i)'\}.$$

Appropriate choices of $\pi_1$ and $\pi_0$ are made in Section 2 and the resultant Bayes tests are derived.

**2. Admissible Bayes tests of (1.1).** Under the alternative $H_1$, we choose $\pi_1$ as the absolutly continuous measure on the space $E^{pk}$ of the $\mu_i$'s given by

$$(2.1)\quad d\pi_1(\mu_1,\cdots,\mu_k)/d\mu_1\cdots d\mu_k =$$

$$(2\pi)^{-kp/2}\prod_{i=1}^{k}|A_i|^{-1/2}(\prod_{i=1}^{k} n_i)^{p/2} etr\{-1/2\sum_{i=1}^{k} n_i A_i^{-1}(\mu_i - \varsigma_i)(\mu_i - \varsigma_i)'\}$$

where $A_i : p\times p$, $p.d.$ matrix, and $\varsigma_i : p\times 1$ vector, $i = 1,\cdots,k$. Clearly $\pi_1$ corresponds to choosing independent normal priors for each $\mu_i$. The matrices $A_i$ and the vectors $\varsigma_i$, $i = 1,\cdots,k$, will be chosen later. This immediately results in the numerator of the expression in (1.4) as

$$(2.2)\qquad \int f^*(\mathbf{x};\theta)\pi_1(d\theta) =$$

$$(2\pi)^{-pk/2}(\prod_{i=1}^{k} n_i)^{p/2}\prod_{i=1}^{k}|A_i|^{-1/2} etr\{-1/2\sum_{i=1}^{k} n_i\Sigma_i^{-1}\bar{\mathbf{x}}_i\bar{\mathbf{x}}_i'\}$$

$$\cdot etr\{-1/2\sum_{i=1}^{k} n_i A_i^{-1}\varsigma_i\varsigma_i'\}$$

$$\cdot etr\{1/2\sum_{i=1}^{k} n_i(\Sigma_i^{-1}\bar{\mathbf{x}}_i + A_i^{-1}\varsigma_i)(\Sigma_i^{-1}\bar{\mathbf{x}}_i + A_i^{-1}\varsigma_i)'(\Sigma_i^{-1} + A_i^{-1})^{-1}\}$$

$$\cdot\int etr\{-1/2\sum_{i=1}^{k} n_i(\Sigma_i^{-1} + A_i^{-1})(\mu_i - (\Sigma_i^{-1} + A_i^{-1})^{-1}$$

$$\cdot(\Sigma_i^{-1}\bar{\mathbf{x}}_i + A_i^{-1}\varsigma_i))(\mu_i - (\Sigma_i^{-1} + A_i^{-1})^{-1}(\Sigma_i^{-1}\bar{\mathbf{x}}_i + A_i^{-1}\varsigma_i))'\}$$

$$\cdot d\mu_1\cdots d\mu_k$$

$$= \prod_{i=1}^{k}|\Sigma_i^{-1}A_i + I_p|^{-1/2}etr\{-1/2\sum_{i=1}^{k} n_i A_i^{-1}\varsigma_i\varsigma_i'\}$$

$$\cdot etr\{-1/2\sum_{i=1}^{k} n_i\Sigma_i^{-1}\bar{\mathbf{x}}_i\bar{\mathbf{x}}_i'\}$$

$$\cdot etr\{1/2\sum_{i=1}^{k} n_i(\Sigma_i^{-1}\bar{\mathbf{x}}_i + A_i^{-1}\varsigma_i)(\Sigma_i^{-1}\bar{\mathbf{x}}_i + A_i^{-1}\varsigma_i)'(\Sigma_i^{-1} + A_i^{-1})^{-1}\}$$

The crux of the problem now is to evaluate the denominator of the expression in (1.4), namely $\int f^*(\mathbf{x}:\theta)\pi_0(d\theta)$ for suitable $\pi_0$'s. Towards this end, first note that

$$(2.3)\qquad \begin{aligned} H\mu_i &= \xi \iff \\ \mu_i &= H'(HH')^{-1}\xi + (I_p - H'(HH')^{-1}H)\eta_i \\ &\qquad for\ arbitrary\ \eta_i \in E^p \end{aligned}$$

so that $\Theta_0$ can be rewritten as

$$(2.4)\quad \Theta_0 = \{(\mu_1,\cdots,\mu_k) : \mu_i = H'(HH')^{-1}\xi + (I_p - H'(HH')^{-1}H)\eta_i,$$

$$\eta_i \in E^p, i = 1,\cdots,k,\ \xi,\eta_1,\cdots,\eta_k\ all\ arbitrary,$$

$H : s \times p$ unknown, $rank(H) = s < p$, $s$ known$\}$.

Our choice of $\pi_0$ on $\Theta_0$ corresponds to choosing suitable priors for $H \in \mathcal{H} = \{H : s \times p,\ rank(H) = s\}$, $\xi \in E^s$ and $\eta_i \in E^p$, $i = 1, \cdots, k$. Throughout the following, as far as $H$ is concerned, we assume that $\pi_0$ assigns all its measure to the subset $\mathcal{H}_1 = \{H : s \times p,\ H = (I_s : \mathbf{0})\}$ of $\mathcal{H}$. It remains to specify priors of $\xi$ and $\eta_1, \cdots, \eta_k$. This is done below.

Choose the conditional density of $(\eta_1, \cdots, \eta_k)$, given $\xi$, as

$$(2.5)\qquad d\pi_0(\eta_1, \cdots, \eta_k|\xi)/d\eta_1 \cdots d\eta_k = (2\pi)^{-kp/2} \prod_{i=1}^{k} |(1/n_i)P_i|^{-1/2}$$
$$etr\{-1/2 \sum_{i=1}^{k} n_i P_i^{-1}(\eta_i - (\eta_i^0 + \Delta_i \xi))(\eta_i - (\eta_i^0 + \Delta_i \xi))'\}$$

and the marginal density of $\xi$ as

$$(2.6)\qquad d\pi_0(\xi)/d\xi = (2\pi)^{-s/2} |Q|^{-1/2} etr\{-1/2 Q^{-1}(\xi - \xi^0)(\xi - \xi^0)'\}$$

where $\eta_i^0 : p \times 1$, $\xi^0 : s \times 1$, $\Delta_i : p \times s$, $P_i : p \times p$, $p.d.$, $Q : s \times s$, $p.d.$ are to be suitably chosen. Let

$$\beta_i = \Sigma_i^{-1} \bar{\mathbf{x}}_i,\ i = 1, \cdots, k,$$
$$\beta = \sum_{i=1}^{k} n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i = \sum_{i=1}^{k} n_i \beta_i,$$
$$\Psi_1 = (I_s : \mathbf{0}) : s \times p.$$

$$
(2.7) \qquad
\begin{aligned}
\Psi_2 &= \begin{pmatrix} 0 & 0 \\ 0 & I_{p-s} \end{pmatrix} : p \times p \\
\Psi_3 &= \Psi_1 \Big(\sum_{i=1}^{k} n_i \Sigma_i^{-1}\Big) \Psi_1' : s \times s, \\
\Psi_{4i} &= \Psi_2 \Sigma_i^{-1} \Psi_2 : p \times p, \ i = 1, \cdots, k, \\
\Psi_{5i} &= \Psi_2 \Sigma_i^{-1} \Psi_1' : p \times s, \ i = 1, \cdots, k.
\end{aligned}
$$

Then the denominator of the expression in (1.4) can be simplified as

$$
\begin{aligned}
(2.8) \ \int f^*(\mathbf{x}; \theta) \pi_0(d\theta) = \\
& (2\pi)^{-pk/2} \Big(\prod_{i=1}^{k} n_i\Big)^{p/2} \prod_{i=1}^{k} |P_i|^{-1/2} (2\pi)^{-s/2} |Q|^{-1/2} \\
& \cdot etr\Big\{-1/2 \sum_{i=1}^{k} n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i'\Big\} etr\{-1/2 Q^{-1} \xi^0 \xi^{0'}\} \\
& \cdot etr\Big\{-1/2 \sum_{i=1}^{k} n_i P_i^{-1} \eta_i^0 \eta_i^{0'}\Big\} \\
& \cdot \int etr\{-1/2 (\Psi_3 + Q^{-1}) \xi \xi' + 1/2 (\Psi_1 \beta + Q^{-1} \xi^0) \xi' \\
& \qquad + 1/2 \xi (\Psi_1 \beta + Q^{-1} \xi^0)'\} \\
& \cdot etr\Big\{-1/2 \sum_{i=1}^{k} n_i [(\Psi_{4i} + P_i^{-1}) \eta_i \eta_i' - (\Psi_2 \beta_i - \Psi_{5i} \xi + \\
& \quad P_i^{-1} (\eta_i^0 + \Delta_i \xi)) \eta_i' - \eta_i (\Psi_2 \beta_i - \Psi_{5i} \xi + P_i^{-1} (\eta_i^0 + \Delta_i \xi))'\Big\} \\
& \cdot etr\Big\{-1/2 \sum_{i=1}^{k} n_i P_i^{-1} (\eta_i^0 \xi' \Delta_i' + \Delta_i \xi \eta_i^{0'} + \Delta_i \xi \xi' \Delta_i')\Big\} \\
& \cdot d\xi d\eta_1 \cdots d\eta_k
\end{aligned}
$$

$$
\begin{aligned}
= \; & (2\pi)^{-pk/2}\Big(\prod_{i=1}^{k} n_i\Big)^{p/2}\prod_{i=1}^{k}|P_i|^{-1/2}(2\pi)^{-s/2}|Q|^{-1/2} \\
& \cdot etr\{-1/2\sum_{i=1}^{k} n_i\Sigma_i^{-1}\bar{\mathbf{x}}_i\bar{\mathbf{x}}_i'\}etr\{-1/2Q^{-1}\xi^0\xi^{0\prime}\} \\
& \cdot etr\{-1/2\sum_{i=1}^{k} n_i P_i^{-1}\eta_i^0\eta_i^{0\prime}\} \\
& \cdot etr\{1/2\sum_{i=1}^{k} n_i(\Psi_2\beta_i + P_i^{-1}\eta_i^0)(\Psi_2\beta_i + P_i^{-1}\eta_i^0)' \\
& \cdot(\Psi_{4i} + P_i^{-1})^{-1}\}etr\{1/2\tau\tau'\Xi^{-1}\} \\
& \cdot\int etr\{-1/2\sum_{i=1}^{k} n_i(\Psi_{4i} + P_i^{-1})(\eta_i - (\Psi_{4i} + P_i^{-1})^{-1}(\Psi_2\beta_i - \Psi_{5i}\xi + P_i^{-1} \\
& \cdot(\eta_i^0 + \Delta_i\xi)))(\eta_i - (\Psi_{4i} + P_i^{-1})^{-1}(\Psi_2\beta_i - \Psi_{5i}\xi + P_i^{-1}(\eta_i^0 + \Delta_i\xi)))'\} \\
& \cdot etr\{-1/2\Xi(\xi - \Xi^{-1}\tau)(\xi - \Xi^{-1}\tau)'\} \\
& \cdot d\xi d\eta_1\cdots d\eta_k \\
= \; & \prod_{i=1}^{k}|P_i\Psi_{4i} + I_p|^{-1/2}|Q|^{-1/2}|\Xi|^{-1/2}etr\{-1/2Q^{-1}\xi^0\xi^{0\prime}\} \\
& \cdot etr\{-1/2\sum_{i=1}^{k} n_i\Sigma_i^{-1}\bar{\mathbf{x}}_i\bar{\mathbf{x}}_i'\}etr\{-1/2\sum_{i=1}^{k} n_i P_i^{-1}\eta_i^0\eta_i^{0\prime}\}etr\{1/2\tau\tau'\Xi^{-1}\} \\
& \cdot etr\{1/2\sum_{i=1}^{k} n_i(\Psi_2\beta_i + P_i^{-1}\eta_i^0)(\Psi_2\beta_i + P_i^{-1}\eta_i^0)'(\Psi_{4i} + P_i^{-1})^{-1}\}
\end{aligned}
$$

where

$$
\begin{aligned}
(2.9) \qquad \Xi = \; & \Psi_3 + Q^{-1} + \sum_{i=1}^{k} n_i\Delta_i'P_i^{-1}\Delta_i \\
& - \sum_{i=1}^{k} n_i(P_i^{-1}\Delta_i - \Psi_{5i})'(\Psi_{4i} + P_i^{-1})^{-1}(P_i^{-1}\Delta_i - \Psi_{5i})
\end{aligned}
$$

and

$$(2.10)\quad \tau = \Psi_1\beta + Q^{-1}\xi^0 - \sum_{i=1}^{k} n_i\Delta_i' P_i^{-1}\eta_i^0 + \sum_{i=1}^{k} n_i(P_i^{-1}\Delta_i - \Psi_{5i})'(\Psi_{4i} + P_i^{-1})^{-1}(\Psi_2\beta_i + P_i^{-1}\eta_i^0).$$

In the above $\Delta_i$'s, $P_i$'s, and $Q$ are chosen so that $\Xi$ is *p.d.*. Hence, combining (2.2) and (2.8), and taking the logarithm of the ratio of the expressions in (2.2) and (2.8), the Bayes test turns out to be of the form

$$(2.11)\quad \begin{aligned} & tr\{\sum_{i=1}^{k} n_i(\Sigma_i^{-1}\bar{\mathbf{x}}_i + A_i^{-1}\varsigma_i)(\Sigma_i^{-1}\bar{\mathbf{x}}_i + A_i^{-1}\varsigma_i)'(\Sigma_i^{-1} + A_i^{-1})^{-1}\} \\ - & tr\{\sum_{i=1}^{k} n_i(\Psi_2\beta_i + P_i^{-1}\eta_i^0)(\Psi_2\beta_i + P_i^{-1}\eta_i^0)'(\Psi_{4i} + P_i^{-1})^{-1}\} \\ - & tr\{\tau\tau'\Xi^{-1}\} \\ > & c'. \end{aligned}$$

We now choose $P_i$, $Q$, $A_i$, $\Delta_i$, $\xi^0$, $\eta_i^0$, and $\varsigma_i$ suitably subject to $\Xi$ being *p.d.* to reduce (2.11) to a nice form.

Let $\Sigma_i$ be partitioned as

$$(2.12)\quad \Sigma_i = \begin{pmatrix} \Sigma_{i(11)} & \Sigma_{i(12)} \\ \Sigma_{i(21)} & \Sigma_{i(22)} \end{pmatrix},$$

where $\Sigma_{i(11)} : s \times s, \Sigma_{i(22)} : (p-s) \times (p-s)$, and let $\Sigma_i^{-1}$ be similarly partitioned into

$$(2.13)\qquad \Sigma_i^{-1} = \begin{pmatrix} \Sigma_i^{11} & \Sigma_i^{12} \\ \Sigma_i^{21} & \Sigma_i^{22} \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma_{i(11\cdot 2)}^{-1} & -\Sigma_{i(11)}^{-1}\Sigma_{i(12)}\Sigma_{i(22\cdot 1)}^{-1} \\ -\Sigma_{i(22)}^{-1}\Sigma_{i(21)}\Sigma_{i(11\cdot 2)}^{-1} & \Sigma_{i(22\cdot 1)}^{-1} \end{pmatrix}$$

where $\Sigma_{i(11\cdot 2)} = \Sigma_{i(11)} - \Sigma_{i(12)}\Sigma_{i(22)}^{-1}\Sigma_{i(21)}$ and $\Sigma_{i(22\cdot 1)} = \Sigma_{i(22)} - \Sigma_{i(21)}\Sigma_{i(11)}^{-1}\Sigma_{i(12)}$, $i = 1, \cdots, k$.

Choose

(i) $\quad P_i = \begin{pmatrix} P_{i(11)} & 0 \\ 0 & \Sigma_{i(22\cdot 1)} \end{pmatrix}, \ i = 1, \cdots, k,$

*where $P_{i(11)}$ is any $s \times s$, p.d. matrix;*

(ii) $\quad Q = (\sum_{i=1}^{k} n_i \Sigma_{i(11)}^{-1})^{-1};$

(iii) $\quad A_i = \begin{pmatrix} \Sigma_{i(11)} & \Sigma_{i(12)} \\ -\Sigma_{i(21)} & \Sigma_{i(22)} \end{pmatrix}, \ i = 1, \cdots, k;$

(iv) $\quad \Delta_i = P_i \Psi_{5i}, \ i = 1, \cdots, k;$

(v) $\quad \xi^0 = Q(\sum_{i=1}^{k} n_i \Psi_{5i}' \eta_i^0);$

(vi) $\quad \zeta_i = A_i(\Sigma_i^{-1} + A_i^{-1})\Psi_2(\Psi_{4i} + P_i^{-1})^{-1}P_i^{-1}\eta_i^0, \ i = 1, \cdots, k.$

and (vii) $\quad \eta_i^0$, $i = 1, \cdots, k$, *arbitray.*

It may be noted that $\Xi$ simplies to

$$
\begin{aligned}
(2.14)\ \Xi &= \Psi_3 + Q^{-1} + \sum_{i=1}^{k} n_i \Psi_{5i}' P_i \Psi_{5i} \\
&= \Psi_1 (\sum_{i=1}^{k} n_i \Sigma_i^{-1}) \Psi_1' + \sum_{i=1}^{k} n_i (\Sigma_{i(11)}^{-1} + \Psi_1 \Sigma_i^{-1} \Psi_2 P_i \Psi_2 \Sigma_i^{-1} \Psi_1') \\
&= \Psi_1 (\sum_{i=1}^{k} n_i \Sigma_i^{-1}) \Psi_1' + \sum_{i=1}^{k} n_i (\Sigma_{i(11)}^{-1} + \Sigma_i^{12} \Sigma_{i(22\cdot 1)} \Sigma_i^{21}) \\
&= \sum_{i=1}^{k} n_i (\Sigma_i^{11} + \Sigma_{i(11)}^{-1} + \Sigma_i^{11} - \Sigma_{i(11)}^{-1}) \\
&= 2 \sum_{i=1}^{k} n_i \Sigma_i^{11}
\end{aligned}
$$

which is clearly $p.d.$ and $P_{i(11)}$ does not affect the calculation.

The three terms appearing in the left hand side of (2.11) can now be evaluated. Combining the first term and the second term of (2.11), and using $(vi)$, we have

$$
\begin{aligned}
(2.15)\quad & tr\{\sum_{i=1}^{k} n_i \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i' \Sigma_i^{-1} ((\Sigma_i^{-1} + A_i^{-1})^{-1} - \Psi_2 (\Psi_{4i} + P_i^{-1})^{-1} \Psi_2) \Sigma_i^{-1} \\
& + \sum_{i=1}^{k} n_i \bar{\mathbf{x}}_i (\varsigma_i' A_i^{-1} (\Sigma_i^{-1} + A_i^{-1})^{-1} - \eta_i^{0\prime} P_i^{-1} (\Psi_{4i} + P_i^{-1})^{-1} \Psi_2) \Sigma_i^{-1} \\
& + \sum_{i=1}^{k} n_i \Sigma_i^{-1} ((\Sigma_i^{-1} + A_i^{-1})^{-1} A_i^{-1} \varsigma_i - \Psi_2 (\Psi_{4i} + P_i^{-1})^{-1} P_i^{-1} \eta_i^{0}) \bar{\mathbf{x}}_i' \\
& + \sum_{i=1}^{k} n_i (\varsigma_i \varsigma_i' A_i^{-1} (\Sigma_i^{-1} + A_i^{-1})^{-1} A_i^{-1} - \eta_i^{0} \eta_i^{0\prime} P_i^{-1} (\Psi_{4i} + P_i^{-1})^{-1} P_i^{-1})\} \\
= \quad & tr\{\sum_{i=1}^{k} n_i \bar{\mathbf{x}}_i \bar{\mathbf{x}}_i' \Sigma_i^{-1} ((\Sigma_i^{-1} + A_i^{-1})^{-1} - \Psi_2 (\Psi_{4i} + P_i^{-1})^{-1} \Psi_2) \Sigma_i^{-1})
\end{aligned}
$$

$$+ \sum_{i=1}^{k} n_i(\varsigma_i\varsigma_i' A_i^{-1}(\Sigma_i^{-1} + A_i^{-1})^{-1}A_i^{-1} - \eta_i^0\eta_i^{0\prime} P_i^{-1}(\Psi_{4i} + P_i^{-1})^{-1}P_i^{-1})\}.$$

Since the second term of the last equation does not contain data, we may drop it. So (2.15) is now

$$(2.16) \quad tr\{\sum_{i=1}^{k} n_i(\Sigma_i^{-1}\bar{\mathbf{x}}_i)(\Sigma_i^{-1}\bar{\mathbf{x}}_i)'((\Sigma_i^{-1} + A_i^{-1})^{-1} - \Psi_2(\Psi_{4i} + P_i^{-1})^{-1}\Psi_2)\}$$

$$= tr\{1/2\sum_{i=1}^{k} n_i(\Sigma_i^{-1}\bar{\mathbf{x}}_i)(\Sigma_i^{-1}\bar{\mathbf{x}}_i)'$$

$$\cdot\left(\begin{pmatrix} \Sigma_{i(11\cdot2)} & 0 \\ 0 & \Sigma_{i(22\cdot1)} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_{i(22\cdot1)} \end{pmatrix}\right)\}$$

$$= tr\left\{1/2\sum_{i=1}^{k} n_i(\Sigma_i^{-1}\bar{\mathbf{x}}_i)(\Sigma_i^{-1}\bar{\mathbf{x}}_i)'\begin{pmatrix} \Sigma_{i(11\cdot2)} & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

Also, using $(ii), (iv)$ and $(v)$, the third term of (2.11) reduces to

$$(2.17) \quad tr\{\tau\tau'\Xi^{-1}\}$$

$$= tr\{1/2(\Psi_1\beta)(\Psi_1\beta)'(\sum_{i=1}^{k} n_i\Sigma_i^{11})^{-1}\}$$

$$= tr\left\{1/2\beta\beta'\begin{pmatrix} (\sum_{i=1}^{k} n_i\Sigma_i^{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right\}$$

$$= tr\left\{1/2(\sum_{i=1}^{k} n_i\Sigma_i^{-1}\bar{\mathbf{x}}_i)(\sum_{i=1}^{k} n_i\Sigma_i^{-1}\bar{\mathbf{x}}_i)'\begin{pmatrix} (\sum_{i=1}^{k} n_i\Sigma_i^{11})^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right\}.$$

Clearly $P_{i(11)}$ does not affect the above calculations. Combining (2.16) and (2.17), (2.11) is now of the form

$$
\begin{aligned}
(2.18) \qquad & tr\left\{\sum_{i=1}^{k} n_i(\Sigma_i^{-1}\bar{\mathbf{x}}_i)(\Sigma_i^{-1}\bar{\mathbf{x}}_i)' \begin{pmatrix} \Sigma_{i(11\cdot 2)} & 0 \\ 0 & 0 \end{pmatrix}\right\} \\
& - \ tr\left\{(\sum_{i=1}^{k} n_i\Sigma_i^{-1}\bar{\mathbf{x}}_i)(\sum_{i=1}^{k} n_i\Sigma_i^{-1}\bar{\mathbf{x}}_i)' \begin{pmatrix} (\sum_{i=1}^{k} n_i\Sigma_{i(11\cdot 2)}^{-1})^{-1} & 0 \\ 0 & 0 \end{pmatrix}\right\} \\
& > \ c''.
\end{aligned}
$$

This immediately prove the following result.

THEOREM 2.1. *An admissible Bayes test of (1.1) rejects $H_0$ for large values of the test statistic given by the left hand side of (2.18).*

When $\Sigma_1, \cdots, \Sigma_k$ are equal (and $= \Sigma$, say), (2.18) reduces to

$$
\begin{aligned}
(2.19) \quad & tr\left\{(\sum_{i=1}^{k} n_i\bar{\mathbf{x}}_i\bar{\mathbf{x}}_i' - 1/n(\sum_{i=1}^{k} n_i\mathbf{x}_i)(\sum_{i=1}^{k} n_i\bar{\mathbf{x}}_i'))\left(\Sigma^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}\right)\right\} \\
& > \ c,
\end{aligned}
$$

which is the same as

$$
(2.20) \qquad tr\left\{B\left(\Sigma^{-1} - \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}\right)\right\} > c \quad ,
$$

where $B$ is defined earlier.

This leads to the following important result.

THEOREM 2.2. *An admissible Bayes test of (1.1) when* $\Sigma_1 = \cdots = \Sigma_k = \Sigma$ *rejects* $H_0$ *for large values of the test statisitc given by the left hand side of (2.19) or (2.20).*

REMARK 2.1. It is interesting to observe that the test statistic given by the left hand side of (2.19) can be interpreted as the sum of the first $s$ roots of the matrix $B\Sigma^{-1}$. This follows because one can always take $\Sigma = I_p$ due to the nature of the hypotheses (1.1) and change $B$ to $\Sigma^{-1/2}B\Sigma^{-1/2}$.

REMARK 2.2. It is clear from the preceding calculations that if, under $H_0, \pi_0$ is chosen as assigning all its measure to the subspace $\Psi_{i_1 \cdots i_s}$ of the form $\Psi_{i_1 \cdots i_s} = \{H : s \times p\ H = (\mathbf{0} \cdots \mathbf{1}_{i_1} \mathbf{0} \cdots \mathbf{1}_{i_2} \cdots \mathbf{1}_{i_s} \cdots \mathbf{0}), \mathbf{1}_{i_j} = (0 \cdots 1 \cdots 0)' : s \times 1$ with 1 at the $j$th position*, $1 \le i_1 < \cdots < i_s \le p\}$, and appropriate changes are made accordingly in the definitions of $\Psi_1, \Psi_2$, and also in the choice of $P_i, Q, A_i$ *etc.*, in the above priors of $\xi$ and $\eta_1, \cdots, \eta_k$, then the Bayes test rejects $H_0$ for large values of the test statistic

$$(2.21) \qquad tr\left\{ \sum_{i=1}^{k} n_i (\Sigma_i^{-1}\bar{\mathbf{x}}_i)(\Sigma_i^{-1}\bar{\mathbf{x}}_i)' \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Sigma_{i(11 \cdot 2)} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$



* $1 \le j \le s$, and $1_{i_j}$ as the $i_j$th column of H,

$$- \; tr \left\{ \left(\sum_{i=1}^{k} n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i\right)\left(\sum_{i=1}^{k} n_i \Sigma_i^{-1} \bar{\mathbf{x}}_i\right)' \begin{pmatrix} 0 & 0 & 0 \\ 0 & \left(\sum_{i=1}^{k} n_i \Sigma_{i(11\cdot 2)}^{-1}\right)^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

where $\Sigma_{i(11\cdot 2)}$ is the appropriate $s \times s$ proper submatrix of $\Sigma_i$ corresponding to the rows $(i_1, \cdots, i_s)$ and the columns $(i_1, \cdots, i_s)$. In the special case of the equality of $\Sigma_1, \cdots, \Sigma_k$, it therefore follows from (2.21) that a test which rejects $H_0$ for large values of the sum of any $s$ roots of $B\Sigma^{-1}$ is admissible Bayes. In particular, the likelihood ratio test of Rao(1973) for equal covariance matrices which rejects $H_0$ for large values of the sum of $s$ smallest roots of $B\Sigma^{-1}$ is admissible Bayes.

## REFERENCES


FISHER, R.A. (1939). The sampling distribution of some statistics obtained from nonlinear equations. *Annals. of Eugenics*, **9**, 238-249.

FUJIKOSHI, Y. (1974). The likelihood ratio tests for the dimensionality of regression coefficients. *J. Mult. Anal.*, **4**, 327-340.

KIEFER, J. and SCHWARTZ, R. (1965). Admissible Bayes character of $T^2$, $R^2$, and other fully invariant tests for classical multivariate normal problems. *Ann. Math. Statist.*, **36**, 747-770.

RAO, C. R. (1973). *Linear Statistical Inference and Its Applications.* 2nd. edition. Wiley, New York.

RAO, C. R. (1985). Tests for dimensionality and interactions of mean vectors under general and reducible covariance structures. *J. Mult. Anal.*, **16**, 173-184.



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